Ring Extensions

S - commutative ring

$R \subset S$ a subring

$\{\alpha_1, \dots, \alpha_n\} \subset S$

Ring Extension of R by $\{\alpha_1, \dots, \alpha_n\}$

Definition

square bracket

f evaluated at $(\alpha_1, \dots, \alpha_n)$

$$R[\alpha_1, \dots, \alpha_n] := \{ f(\alpha_1, \dots, \alpha_n) \mid f \in R[x_1, \dots, x_n] \}$$

Exercise: $R[\alpha_1, \dots, \alpha_n] \subset S$ is a subring

minimal subring of S containing R and $\{\alpha_1, \dots, \alpha_n\}$

Remarks

1) S an integral domain $\Rightarrow$ $R[\alpha_1, \dots, \alpha_n]$ integral domain

2) $\mathrm{Char}(R[\alpha_1, \dots, \alpha_n]) = \mathrm{Char}(S)$

S an integral domain

Examples

1) $\{\alpha_1, \dots, \alpha_n\} \subset R \Rightarrow R[\alpha_1, \dots, \alpha_n] = R$

2) $R = \mathbb{Z}$, $S = \mathbb{C}$. $\mathbb{Z}[i] :=$ Gaussian Integers

$i^2 = -1 \Rightarrow \forall f \in \mathbb{Z}[x]$, $f(i) = a + bi$ for some $a, b \in \mathbb{Z}$ $\Rightarrow$ $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$

3) $R = \mathbb{Z}$, $S = \mathbb{Q}$

$$\mathbb{Z}[\tfrac{1}{n}] = \left\{ \frac{a}{n^k} \mid a \in \mathbb{Z}, k \in \mathbb{N} \cup \{0\} \right\} \subset \mathbb{Q}$$

E - field

$F \subset E$ a subfield

subring which is also a field

$\{\alpha_1, \dots, \alpha_n\} \subset E$

<u>Definition</u>

Field extension of F by $\{\alpha_1, \dots, \alpha_n\}$

$$F(\alpha_1, \dots, \alpha_n) = \left\{ \frac{f(\alpha_1, \dots, \alpha_n)}{g(\alpha_1, \dots, \alpha_n)} \;\middle|\; \begin{array}{l} f, g \in E[x_1, \dots, x_n] \text{ and} \\ g(\alpha_1, \dots, \alpha_n) \neq 0_E \end{array} \right\}$$

Parenthases

$$\frac{f(\alpha_1, \dots, \alpha_n)}{g(\alpha_1, \dots, \alpha_n)} = f(\alpha_1, \dots, \alpha_n)\left(g(\alpha_1, \dots, \alpha_n)\right)^{-1}$$

<u>Exercise</u> $F(\alpha_1, \dots, \alpha_n) \subset E$ is a subfield.

minimal subfield of E containing F and $\{\alpha_1, \dots, \alpha_n\}$

<u>Examples</u>

1/ $F = \mathbb{Q}$, $E = \mathbb{C}$

$$\mathbb{Q}(i) = \left\{ \frac{a+bi}{c+di} \;\middle|\; \begin{array}{l} a, b, c, d \in \mathbb{Q}, \\ c + di \neq 0 \end{array} \right\} \subset \mathbb{C}$$

2/ $F = \mathbb{Q}$, $E = \mathbb{C}$, $\underbrace{\{\sqrt[3]{2}, \sqrt[3]{2}e^{\frac{2\pi i}{3}}, \sqrt[3]{2}e^{\frac{4\pi i}{3}}\}}_{\text{All complex roots of } x^3-2} \subset \mathbb{C}$

$$\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}e^{\frac{2\pi i}{3}}, \sqrt[3]{2}e^{\frac{4\pi i}{3}}) = \text{smallest subfield of } \mathbb{C} \text{ containing all roots of } x^3-2$$

Note $$e^{\frac{2\pi i}{3}} = \frac{\sqrt[3]{2}e^{\frac{2\pi i}{3}}}{\sqrt[3]{2}} \in \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}e^{\frac{2\pi i}{3}}, \sqrt[3]{2}e^{\frac{4\pi i}{3}})$$

$$\Rightarrow \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}e^{\frac{2\pi i}{3}}, \sqrt[3]{2}e^{\frac{4\pi i}{3}}) = \mathbb{Q}(\sqrt[3]{2}, e^{\frac{2\pi i}{3}})$$

can sometimes simplify generating set.